Question: Use eigenvalues and eigenvectors to find the solution to the following IVP:

$$\mathbf{x}' = \begin{pmatrix} -2 & 5 \\ -1 & -4 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Solution:
The eigenvalues are found by solving the characteristic polynomial equation $|A - I\lambda| = 0$ of the matrix

$$A = \begin{bmatrix} -2 & 5 \\ -1 & 4 \end{bmatrix}.$$

This results in

$$(-2 - \lambda)(-4\lambda) - (-1)(5) = 0$$

which has solutions

$$\lambda_1 = -3 + 2i \text{ and } \lambda_2 = -3 - 2i$$

To find the eigenvectors corresponding to λ_1 and λ_2, we substitute into $A - I\lambda$ and solve $(A - I\lambda_n)\mathbf{v}_n = \mathbf{0}$ to obtain

$$\mathbf{v}_1 = \begin{bmatrix} -1 - 2i \\ 1 \end{bmatrix} \text{and } \mathbf{v}_2 = \begin{bmatrix} -1 + 2i \\ 1 \end{bmatrix}.$$

Accordingly, the general solution can be constructed as

$$\mathbf{x}(t) = C_1\mathbf{v}_1 e^{\lambda_1 t} + C_2\mathbf{v}_2 e^{\lambda_2 t}$$

$$= C_1 \begin{bmatrix} -1 - 2i \\ 1 \end{bmatrix} e^{(-3+2i)t} + C_2 \begin{bmatrix} -1 + 2i \\ 1 \end{bmatrix} e^{(-3-2i)t}$$

Applying Euler's identity

$$= C_1 \begin{bmatrix} (-1 - 2i)(\cos 2t + i\sin 2t) \\ \cos 2t + i\sin 2t \end{bmatrix} e^{-3t} + C_2 \begin{bmatrix} (-1 + 2i)(\cos 2t - i\sin 2t) \\ \cos 2t - i\sin 2t \end{bmatrix} e^{-3t}$$

and collecting real and imaginary terms gives

$$= (C_1 + C_2) \begin{bmatrix} -\cos 2t + 2\sin 2t \\ \cos 2t \end{bmatrix} e^{-3t} + (C_1 - C_2)i \begin{bmatrix} -2\cos 2t - \sin 2t \\ \sin 2t \end{bmatrix} e^{-3t}$$

$$= A \begin{bmatrix} -\cos 2t + 2\sin 2t \\ \cos 2t \end{bmatrix} e^{-3t} + B \begin{bmatrix} -2\cos 2t - \sin 2t \\ \sin 2t \end{bmatrix} e^{-3t}$$

where we have defined

$$A = C_1 + C_2 \text{ and } B = (C_1 - C_2)i.$$

Applying the initial condition $\mathbf{x}(0) = [3, 1]^T$ to this equation gives

$$A\begin{bmatrix}-1\\1\end{bmatrix} + B\begin{bmatrix}-2\\0\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$

which has solution

$$\{A = 1, B = -2\}$$

Finally, we substitute these values into our solution and simplify to obtain the final answer

$$\mathbf{x}(t) = \begin{bmatrix}3\cos 2t + 4\sin 2t\\ \cos 2t - 2\sin 2t\end{bmatrix} e^{-3t}$$

$$= \boxed{e^{-3t}\cos(2t)\begin{bmatrix}3\\1\end{bmatrix} + e^{-3t}\sin(2t)\begin{bmatrix}-4\\2\end{bmatrix}}$$

or, in matrix-product form,

$$= \begin{bmatrix}-3 & -4\\1 & 2\end{bmatrix}\begin{bmatrix}\cos(2t)\\ \sin(2t)\end{bmatrix} e^{-3t}$$

The solution is plotted for $t > 0$. The color of the vector field indicates rate of change with red being faster than blue.